Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Approximation Theory 120 (2003) 217–241
JOURNAL OF Approximation Theory
http://www.elsevier.com/locate/jat

L_∞ convergence of interpolation and associated product integration for exponential weights

H.S. Jung

Division of Applied Mathematics, KAIST, 373-1 Kusongdong, Yusongku, Taejon 305-701, South Korea

Received 7 February 2002; accepted in revised form 27 September 2002

Abstract

We investigate convergence in a weighted L_∞ norm of Hermite–Fejér, Hermite, and Grünwald interpolations at zeros of orthogonal polynomials with respect to exponential weights such as Freud, Erdős, and exponential weight on $(-1,1)$. Convergence of product integration rules induced by the various approximation processes is deduced. We also give more precise weight conditions for convergence of interpolations with respect to above three types of weights, respectively.

Keywords: Hermite; Hermite–Fejér; Grünwald interpolations; Product integration rules; Exponential weights

1. Introduction

For a function $f:(a,b)\to\mathbb{R}, -\infty\leqslant a<b\leqslant\infty$ and a set

$$\chi_n := \{x_{1n}, x_{2n}, \dots, x_{nn}\}, \quad n\geqslant 1$$

of pairwise distinct nodes let $H_n[\chi_n;f]$ and $\hat{H}_n[\chi_n;f]$ denote the Hermite–Fejér and Hermite interpolation polynomials of degree $\leqslant 2n-1$ to f with respect to χ_n. For the case of Hermite interpolation, we will always assume that f is differentiable so that $\hat{H}_n[\chi_n;f]$ is well defined. In fact, $H_n[\chi_n;f]$ and $\hat{H}_n[\chi_n;f]$ are the unique polynomials of degree $\leqslant 2n-1$ satisfying:

$$\begin{cases} H_n[\chi_n;f](x_{jn}) = f(x_{jn}), \\ H_n'[\chi_n;f](x_{jn}) = 0, \end{cases} \quad \begin{cases} \hat{H}_n[\chi_n;f](x_{jn}) = f(x_{jn}), \\ \hat{H}_n'[\chi_n;f](x_{jn}) = f'(x_{jn}) \end{cases} \tag{1.1}$$

for $j=1,2,\dots,n$.

E-mail address: hsjung@jacobi.kaist.ac.kr.

0021-9045/02/$ - see front matter
PII: S0021-9045(02)00020-5

Here, we are interested in L_∞ convergence of Hermite–Fejér and Hermite interpolations with respect to χ_n whose elements are the zeros of a sequence of orthogonal polynomials. More precisely, in this paper we consider $w(x) := \exp(-Q(x))$, where $Q: I \to \mathbb{R}$ is even, continuous, and of at least polynomial growth at the end of interval I and I is either $(-1,1)$ or $\mathbb{R}$. Then χ_n consists of the zeros $\{x_{j,n}(w^2)\}_{j=1}^n$ of the nth orthonormal polynomial $p_n(w^2,x)$

$$p_n(w^2,x) = \gamma_n(w^2)x^n + \text{lower degree terms } (\gamma_n(d\alpha)>0)$$

with respect to w^2, defined by the condition

$$\int_I p_n(w^2,x)p_m(w^2,x)w^2(x) = \delta_{mn}, \quad m,n = 0,1,2,\ldots \; .$$

Then all $\{x_{j,n}(w^2)\}_{j=1}^n$ belongs to I, which we arrange as

$$x_{n,n}(w^2) < x_{n-1,n}(w^2) < \cdots < x_{2,n}(w^2) < x_{1,n}(w^2).$$

Let $H_n[w^2;\cdot]$ and $\hat{H}_n[w^2;\cdot]$ be the Hermite–Fejér and Hermite interpolation operators with respect to the zeros $\{x_{j,n}(w^2)\}_{j=1}^n$ of $p_n(w^2;x)$. Then by (1.1), we have, (cf. [16, p. 330])

$$H_n[w^2;f](x) = H_{n1}[w^2;f](x) + H_{n2}[w^2;f](x), \tag{1.2}$$

where

$$H_{n1}[w^2;f](x) := \sum_{k=1}^n f(x_{kn})l_{kn}^2(x),$$

$$H_{n2}[w^2;f](x) := \sum_{k=1}^n -\frac{p_n''(x_{kn})}{p_n'(x_{kn})}(x - x_{kn})f(x_{kn})l_{kn}^2(x)$$

and

$$\hat{H}_n[w^2;f](x) = H_n[w^2;f](x) + H_{n3}[w^2;f'](x), \tag{1.3}$$

where

$$H_{n3}[w^2;f'](x) := \sum_{k=1}^n f'(x_{kn})(x - x_{kn})l_{kn}^2(x).$$

Here, $l_{kn}(x)$ is the fundamental Lagrange interpolation polynomial [6, p. 23], given by

$$l_{kn}(w^2;x) := \frac{p_n(w^2;x)}{p_n'(w^2;x_{kn})(x - x_{kn})}, \quad k = 1,2,\ldots,n.$$

$H_{n1}[w^2;\cdot]$ is the so-called Grünwald operator, which is positive, that is, $H_{n1}[w^2;f](x) \geqslant 0$ in I when $f \geqslant 0$ in I.

Our main concern is the following problem: Under what conditions on weight functions $w_1(x)$ and $w_2(x)$ will the relation

$$\lim_{n\to\infty} ||(f(x) - H_n[w^2;f](x))w_1(x)||_{L_\infty(I)} = 0 \tag{1.4}$$

hold for all continuous functions f satisfying $\lim_{|x|\to\infty \text{ or } 1} |fw_2(x)| = 0$? Lubinsky [11] proved (1.4) in case

$$w_1(x) = \frac{w^2}{(1+Q'(x))^{\alpha}(1+|x|)}, \quad w_2(x) = w^2(1+Q'(x))^{\beta}(1+|x|)^2$$

and $w(x)$ is a Freud weight or an Erdős weight. Recently, Szabados [15] has proved (1.4) in case

$$w_1(x) = \frac{w^2}{(1+|x|)^{1/3}}, \quad w_2(x) = w^2(1+Q'(x))$$

and $w(x)$ is a Freud weight. In this paper, we will extend Szabados' result to a class of Freud, Erdős, and exponential weights on $(-1,1)$, and we also find more precise weight conditions for these three types of weights. L_p $(0<p<\infty)$ convergence of Hermite–Fejér and Hermite interpolations, are handled in [3,7] for Freud and Erdős weights.

Once we have the convergence of interpolations, we can consider the convergence of the associated *product quadrature rules*, involving approximation of

$$I[k;f] := \int_I k(x)f(x)\,dx$$

by quadrature rules

$$I_n[k;f] := \sum_{j=1}^{n} w_{jn}(k)f(x_{jn}),$$

where the weights $\{w_{jn}(k)\}$ are usually determined by integration of some approximation to f. Here, the kernel $k(x)$ is typically the difficult component of the integrand $k(x)f(x)$, with known types of singularities or oscillatory behavior and usually f has smooth behavior. The product quadrature treated in this paper, is to approximate $I[k;f]$ by

$$I_n[k;f] := \int_I k(x)H_n[w^2;f]\,dx = \sum_{j=1}^{n} f(x_{jn})\left(\int_I k(x)h_{jn}(x)\,dx\right), \tag{1.5}$$

where

$$h_{jn}(x) := \left\{1 - \frac{p_n''(x_{jn})}{p_n'(x_{jn})}(x-x_{jn})\right\} l_{jn}^2(x).$$

Analogous rules generated by $\hat{H}_n$ and H_{n1} are

$$\begin{aligned} \hat{I}_n[k;f] &:= \int_I k(x)\hat{H}_n[w^2;f]\,dx \quad \text{and} \\ J_n[k;f] &:= \int_I k(x)H_{n1}[w^2;f]\,dx. \end{aligned} \tag{1.6}$$

We shall prove these product quadratures converge to $I[k,f]$ under mild conditions on $f(x)$ and $k(x)$.

This paper is organized as follows. In Section 2, we introduce our admissible class of weights and state main results. In Section 3, we present some technical estimates. In Section 4, we prove the results of Section 2. Finally, in Section 5, we give more precise weight conditions for Freud, Erdős, and exponential weights on $(-1,1)$.

2. Main results

We first introduce some notations, which we use in the following. For any two sequences $\{b_n\}_n$ and $\{c_n\}_n$ of non-zero real numbers (or functions), we write $b_n \lesssim c_n$, if there exists a constant $C>0$, independent of n(and x) such that $b_n \leqslant Cc_n$ for n large enough and write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote by $\mathscr{P}_n$ the space of polynomials of degree at most n. Let $I+$ be either $(0,\infty)$ if $I=\mathbb{R}$ or $(0,1)$ if $I=(-1,1)$.

We now introduce an admissible class of weights.

Definition 2.1. Let $w_Q = \exp(-Q)$ where $Q(x): I \to \mathbb{R}$ is even, continuous, and

(a) $Q''(x)$ is continuous in $I+$ and $Q''(x)$, $Q'(x) \geqslant 0$ in $I+$;
(b) the function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \quad x \in I+$$

satisfies for large enough x or x close enough to ± 1

$$T(x) \sim \frac{xQ'(x)}{Q(x)}. \tag{2.1}$$

Moreover, T satisfies either

(b1) There exist $A>1$ and $B>1$ such that

$$A \leqslant T(x) \leqslant B, x \in \mathbb{R}+.$$

(b2) T is increasing in $I+$ with $\lim_{x\to 0+} T(x) > 1$. If $I=\mathbb{R}$,

$$\lim_{|x|\to\infty} T(x) = \infty$$

and if $I=(-1,1)$, for x close enough to ± 1 and some $A>2$,

$$T(x) \geqslant \frac{A}{1-x^2}.$$

Then $w(x)$ shall be called an admissible weight and we shall write $w \in \mathscr{A}$.

We let a_u, for $u>0$, be the uth Mhaskar–Rahmanov–Saff number, which is the unique positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)\, dt/\sqrt{1-t^2}. \tag{2.2}$$

Then, a_u is increasing with u. The importance of a_n lies in the identity

$$||Pw||_{L_\infty[I]} = ||Pw||_{L_\infty[-a_n,a_n]}$$

for any polynomial $P \in \mathscr{P}_n$. We define some auxiliary quantities which we will need in the sequel. See [8–10]. Set

$$\delta_n := (nT(a_n))^{-2/3}, \quad n \geqslant 1, \tag{2.3}$$

which are useful in describing the behavior of $p_n(w^2, x)$ near x_{1n}. For example, for $w \in \mathscr{A}$

$$|x_{1n}/a_n - 1| \leqslant \frac{L}{2}\delta_n,$$

where L is a positive constant independent of n. We also need the sequence of functions

$$\Psi_n(x) := \begin{cases} \max\{\sqrt{1-\frac{|x|}{a_n}+L\delta_n}, \frac{1}{T(a_n)\sqrt{1-\frac{|x|}{a_n}+L\delta_n}}\}, & |x| \leqslant a_n, \\ \Psi_n(a_n), & |x| \geqslant a_n, \end{cases} \tag{2.4}$$

which are useful in describing the spacing of zeros of $p_n(w^2, x)$ and Christoffel functions.

The followings are our main results.

Theorem 2.2. *Let $w \in \mathscr{A}$. Let $u(x)$ and $v^{-1}(x)$ be even functions that are non-decreasing on $I \cap (0, \infty)$ and for some $L>0$,*

$$A_n := ||(|1-|x|/a_n| + L\delta_n)^{-1/2} v(x)||_{L_\infty(I)} \times ||s_n(x)u^{-1}(x)||_{L_\infty(|x| \leqslant a_n(1+L\delta_n))} \frac{a_n \log n}{n},$$

where

$$s_n(x) := Q'(x)\Psi_n(x)|1-|x|/a_n + L\delta_n|^{1/2}.$$

Then for a continuous function $f: I \to \mathbb{R}$ and large enough $n \geqslant 1$,

(a) $$||H_{n1}[w^2,f](x)w^2||_{L_\infty(I)} \lesssim ||fw^2||_{L_\infty(I)}. \tag{2.5}$$

(b) $$||H_{n2}[w^2,f](x)w^2 v(x)||_{L_\infty(I)} \lesssim A_n ||fw^2 u||_{L_\infty(I)}. \tag{2.6}$$

(c) $$||H_n[w^2,f](x)w^2 v(x)||_{L_\infty(I)} \lesssim (A_n+1)||fw^2 u||_{L_\infty(I)}. \tag{2.7}$$

(d) *If we let*

$$B_n := ||(|1 - |x|/a_n| + L\delta_n)^{-1/2} v(x)||_{L_\infty(I)} \frac{a_n \log n}{n}$$

then

$$||H_{n3}[w^2, f'](x) w^2 v(x)||_{L_\infty(I)} \lesssim B_n ||f' w^2||_{L_\infty(I)}. \tag{2.8}$$

(e) $$||\hat{H}_n[w^2, f](x) w^2 v(x)||_{L_\infty(I)} \lesssim (A_n + 1)||f w^2 u||_{L_\infty(I)} + B_n ||f' w^2||_{L_\infty(I)}. \tag{2.9}$$

Now, we can extend Szabados' results using Theorem 2.2.

Theorem 2.3. *Let* $w \in \mathscr{A}$, $u(x) := |Q'(x)| + 1$, *and* $v(x) := (|x| + 1)^{-1/3}$.

(a) *For a continuous function f on I satisfying*

$$\lim_{|x| \to \infty \ \ or \ \ 1} |f(x) w^2(x)| = 0, \tag{2.10}$$

$$\lim_{n \to \infty} ||(f(x) - H_{n1}[w^2, f](x)) w^2||_{L_\infty(I)} = 0.$$

(b) *For a continuous function f on I satisfying*

$$\lim_{|x| \to \infty \ \ \text{or} \ \ 1} |f(x) w^2(x) u(x)| = 0, \tag{2.11}$$

$$\lim_{n \to \infty} ||(f(x) - H_n[w^2, f](x)) w^2 v(x)||_{L_\infty(I)} = 0.$$

(c) *For a continuous function f on I satisfying* (2.11) *and* $||f' w^2||_{L_\infty(I)} < \infty$,

$$\lim_{n \to \infty} ||(f(x) - \hat{H}_n[w^2, f](x)) w^2 v(x)||_{L_\infty(I)} = 0.$$

Theorem 2.4. *Let* $w \in \mathscr{A}$ *and*

$$K_{r,n,p}(f, W, t^r) := \inf_{P \in \mathscr{P}_n} \{||(f - P)W||_{L_p(I)} + t^r ||P^{(r)} \Phi_t^r W||_{L_p(I)}\},$$

where $\Phi_t(x) := |1 - |x|/\sigma(t)|^{1/2} + T^{-1/2}(\sigma(t))$ *and* $\sigma(t) := \inf\{a_u : a_u/u \leqslant t\}$. *Then for any continuous function f on I and* $x \in I$, *we have*

$$\begin{aligned} &|(H_n[w^2, f](x) - f(x)) w^2(x)| \\ &\lesssim K_{1,2n-1,\infty}\left(f, w^2, \frac{a_n}{n}\right) \\ &\quad \times \left(1 + (|1 - |x|/a_n| + L\delta_n)^{-1/2} \log n \begin{cases} \Phi_{\frac{a_n}{n}}^{-1}(x), & |x| \leqslant a_{2n} \\ T^{1/2}(a_n), & |x| \geqslant a_{2n} \end{cases}\right). \end{aligned}$$

For the product integration rules I_n, $\hat{I}_n$, and J_n defined by (1.5) and (1.6), we can prove:

Corollary 2.5. *Assume that hypotheses of Theorem* 2.3 *hold. Let* $k: I \to \mathbb{R}$ *be measurable and assume that*

$$\int_I |k(x)| w^{-2} v^{-1}(x)\, dx < \infty.$$

(a) *For a continuous function f on I satisfying* (2.11),

$$\lim_{n\to\infty} I_n[k;f] = I[k;f] := \int_I k(x) f(x)\, dx. \tag{2.12}$$

(b) *For a continuous function f on I satisfying* (2.11) *and* $\|f' w^2\|_{L_\infty(I)} < \infty$,

$$\lim_{n\to\infty} \hat{I}_n[k;f] = I[k;f]. \tag{2.13}$$

(c) *For a continuous function f satisfying* (2.10) *and*

$$\int_I |k(x)| w^{-2}(x)\, dx < \infty,$$

$$\lim_{n\to\infty} J_n[k;f] = I[k;f].$$

3. Lemmas

Convergence of interpolation is closely connected to bounds on orthogonal polynomials and related estimates, which we recall now.

Proposition 3.1. *Let* $w \in \mathscr{A}$.

(a) *For* $n \geqslant 1$,

$$|x_{1n}/a_n - 1| \lesssim \delta_n \tag{3.1}$$

and uniformly for $n \geqslant 2$ *and* $1 \leqslant j \leqslant n-1$,

$$x_{jn} - x_{j+1,n} \sim \frac{a_n}{n} \Psi_n(x_{jn}). \tag{3.2}$$

(b) *For* $n \geqslant 1$,

$$\sup_{x\in I} |p_n(x)| w(x) |1 - |x|/a_n|^{1/4} \sim a_n^{-1/2} \tag{3.3}$$

and

$$\sup_{x\in I} |p_n(x)|w(x) \sim a_n^{-1/2}(nT(a_n))^{1/6}. \tag{3.4}$$

(c) *Uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and* $x \in \mathbb{R}$,

$$|l_{jn}(x)| \sim \frac{a_n^{3/2}}{n}(\Psi_n w)(x_{jn})(1 - |x_{jn}|/a_n + L\delta_n)^{1/4}\left|\frac{p_n(x)}{x - x_{jn}}\right|. \tag{3.5}$$

(d) *Uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and* $x \in \mathbb{R}$,

$$|l_{jn}(x)|w^{-1}(x_{jn})w(x) \lesssim 1. \tag{3.6}$$

(e) *Uniformly for* $u \geqslant C$ *and* $j = 0, 1, 2,$

$$a_u^j Q^{(j)}(a_u) \sim uT(a_u)^{j-1/2}. \tag{3.7}$$

(f) *Let* $0 < \alpha < \beta$. *Then uniformly for* $u \geqslant C$ *and* $j = 0, 1, 2,$

$$T(a_{\alpha u}) \sim T(a_{\beta u}), \quad a_{\alpha u} \sim a_{\beta u}, \quad \text{and} \quad Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u}). \tag{3.8}$$

(g) *Uniformly for* $n \geqslant 2$ *and* $1 \leqslant j \leqslant n-1$,

$$1 - |x_{jn}|/a_n + L\delta_n \sim 1 - |x_{j+1,n}|/a_n + L\delta_n \tag{3.9}$$

and so uniformly for $n \geqslant 2$ *and* $1 \leqslant j \leqslant n-1$

$$\Psi_n(x_{jn}) \sim \Psi_n(x_{j+1,n}).$$

(h) *Given any fixed* $r > 1$,

$$\frac{a_{ru}}{a_u} - 1 \sim \frac{1}{T(a_u)}. \tag{3.10}$$

(i) *Let* $0 < \eta < 1$. *Uniformly for* $n \geqslant 1$, $0 < |x| \leqslant a_{\eta n}$, *and* $|x| = a_s$,

$$C_1 \leqslant T(x)\left(1 - \frac{|x|}{a_n}\right) \leqslant C_2 \log \frac{n}{s}. \tag{3.11}$$

(j) *There exists a constant* ε *with* $0 < \varepsilon < 2$ *such that for* $n \geqslant 1$,

$$T(a_n) \lesssim \left(\frac{n}{a_n}\right)^{2-\varepsilon}. \tag{3.12}$$

(k) *For* $0 < s \leqslant t < 1$,

$$\left(\frac{t}{s}\right)^{T(s)} \leqslant \frac{tQ'(t)}{sQ'(s)} \leqslant \left(\frac{t}{s}\right)^{T(t)}. \tag{3.13}$$

(l) *For* $|x| \leqslant 1$,
$$Q'(a_n x)(1-|x|)^{1/2} \lesssim n/a_n. \tag{3.14}$$

Proof. (a) These follow from Corollary 1.2(a), (b) in [8], Corollary 1.4(i), (1.35) in [9], and Corollary 1.3(a), (b) in [10].

(b) These follow from Corollary 1.4 in [8], Corollary 1.5(i), (ii) in [9], and Corollary 1.4(a) in [10].

(c) It follows from the formula of l_{jn} and Corollary 1.3 in [8], Corollary 1.5(iii) in [9], and Corollary 1.4(b) in [10].

(d) It follows from Lemma 2.6(b) in [13], Lemma 12.2(b) in [9, p. 134], and Theorem 1.2(b) in [12].

(e)–(f) For (b1) case, these follow from (b1) condition, Lemma 5.1(c), and Lemma 5.2(c) in [8]. Otherwise, these follow from part of Lemma 3.2 in [9] and Lemma 2.2 in [10].

(g) These follow from (11.10) in [8, p. 521], (10.12) in [9, p. 111], (9.9) in [10, p. 265], and the definition of Ψ_n.

(h) It follows from Lemma 5.2(c) in [8], Lemma 3.2(v) in [9], and Lemma 2.2(v) in [10].

(i) It follows from the proof of Lemma 2.4(c) in [4].

(j) For (b1) case, since $T(x)$ is bounded, it follows from Lemma 5.2(b) in [8]. Otherwise, it follows from Lemma 3.2(iii) in [9] and Lemma 2.2(viii) in [10].

(k) It follows from Lemma 5.1(b) in [8], Lemma 3.1(i) in [9], and Lemma 2.1(i) in [10].

(l) We may assume $x>0$. Then by (2.2)

$$\frac{n}{a_n} = \frac{2}{\pi}\int_0^1 \frac{tQ'(a_n t)}{\sqrt{1-t^2}}\,dt \geqslant \frac{2}{\pi}\int_x^1 \frac{tQ'(a_n t)}{\sqrt{1-t^2}}\,dt$$

$$\geqslant \frac{2}{\pi}\frac{Q'(a_n x)}{\sqrt{1-x^2}}\int_x^1 t\,dt = \frac{1}{\pi}Q'(a_n x)\sqrt{1-x^2}. \qquad \square$$

Lemma 3.2 (Lubinsky and Rabinowitz [14], Lubinsky [11]). *Let* $w \in \mathscr{A}$. *Uniformly for* $1 \leqslant k \leqslant n$

$$\left|\frac{p_n''(x_{kn})}{p_n'(x_{kn})}\right| \lesssim Q'(x_{kn}).$$

Proof. By (3.7), (3.8), (3.11), and (3.13), it follows from Lemma 4.3 in [11] and Lemma 5.3 in [14]. $\square$

Let $j(x) \in \mathbb{N}$ be defined by

$$|x - x_{j(x),n}| = \min_{1 \leqslant k \leqslant n} |x - x_{kn}|, \quad x \in I.$$

Lemma 3.3 (Damelin [1,2], Szabados [15]). *Let $w \in \mathscr{A}$. Uniformly for $x \in I$ and for $1 \leqslant k \leqslant n$ and $n \geqslant N_0$,*

$$\sum_{\substack{k=1 \\ k \notin [j(x)-2, j(x)+2]}}^{n} \frac{\Delta x_{k,n}}{|x - x_{k,n}|^{\alpha}} = \begin{cases} O(a_n^{1-\alpha}), & 0 < \alpha < 1, \\ O(\log n), & \alpha = 1, \\ O(\frac{n}{a_n \Psi_n(x)})^{\alpha - 1}, & \alpha > 1. \end{cases}$$

Proof. It follows from Lemma 6 in [15, p. 108], Lemma 4.1 in [1, p. 235], and Lemma 3.2 in [2, p. 252]. □

We can now prove main estimates for our results.

Lemma 3.4. *Let $w \in \mathscr{A}$ and $u(x)$ be an even and non-decreasing function.*

(a) *For large enough $n \geqslant 1$ and $x \in I$,*

$$\sum_{k=1}^{n} Q'(x_{kn})|x - x_{kn}| l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) u^{-1}(x_{kn})$$

$$\lesssim (|1 - |x|/a_n| + L\delta_n)^{-1/2} ||s_n(x) u^{-1}||_{L_\infty(|x| \leqslant a_n(1+L\delta_n))} \frac{a_n \log n}{n}.$$

(b) *For large enough $n \geqslant 1$ and $x \in I$,*

$$\sum_{k=1}^{n} l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) \lesssim 1.$$

(c) *For large enough $n \geqslant 1$ and $x \in I$,*

$$\sum_{k=1}^{n} |x - x_{kn}| l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) \lesssim (|1 - |x|/a_n| + L\delta_n)^{-1/2} \frac{a_n}{n} \log n.$$

(d) *For large enough $n \geqslant 1$ and $x \in I$,*

$$\sum_{k=1}^{n} \Phi_{\frac{a_n}{n}}^{-1}(x_{kn}) |x - x_{kn}| l_{kn}^2(x) w^2(x) w^{-2}(x_{kn})$$

$$\lesssim (|1 - |x|/a_n| + L\delta_n)^{-1/2} \frac{a_n}{n} \log n \begin{cases} \Phi_{\frac{a_n}{n}}^{-1}(x), & |x| \leqslant a_{2n}, \\ T^{1/2}(a_n), & |x| \geqslant a_{2n}. \end{cases}$$

Proof. (a) For $x \in I$, by (3.1), (3.2), (3.6), and (3.9)

$$\begin{aligned}
&\sum_{k \in [j(x)-2, j(x)+2] \cap [1,n]} Q'(x_{kn})|x - x_{kn}| l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) u^{-1}(x_{kn}) \\
&\lesssim \sum_{k \in [j(x)-2, j(x)+2] \cap [1,n]} Q'(x_{kn}) \frac{a_n}{n} \Psi_n(x_{kn}) u^{-1}(x_{kn}) \\
&\lesssim \frac{a_n}{n} \|s_n(x) u^{-1}\|_{L_\infty(|x| \leqslant a_n(1+L\delta_n))} \sum_{k \in [j(x)-2, j(x)+2] \cap [1,n]} |1 - |x_{kn}|/a_n + L\delta_n|^{-1/2} \\
&\sim \frac{a_n}{n} \|s_n(x) u^{-1}\|_{L_\infty(|x| \leqslant a_n(1+L\delta_n))} (|1 - |x|/a_n| + L\delta_n)^{-1/2}
\end{aligned}$$

and we have by (3.1)–(3.5), and Lemma 3.3,

$$\begin{aligned}
&\sum_{k \notin [j(x)-2, j(x)+2] \cap [1,n]} Q'(x_{kn})|x - x_{kn}| l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) u^{-1}(x_{kn}) \\
&\lesssim |a_n^{1/2} p_n(x) w(x)|^2 \frac{a_n}{n} \sum s_n(x_{kn}) u^{-1}(x_{kn}) \frac{\Delta x_{kn}}{|x - x_{kn}|} \\
&\lesssim |a_n^{1/2} p_n(x) w(x)|^2 \frac{a_n}{n} \|s_n(x) u^{-1}\|_{L_\infty(|x| \leqslant a_n(1+L\delta_n))} \sum \frac{\Delta x_{kn}}{|x - x_{kn}|} \\
&\lesssim |a_n^{1/2} p_n(x) w(x)|^2 \frac{a_n \log n}{n} \|s_n(x) u^{-1}\|_{L_\infty(|x| \leqslant a_n(1+L\delta_n))} \\
&\lesssim (|1 - |x|/a_n| + L\delta_n)^{-1/2} \frac{a_n \log n}{n} \|s_n(x) u^{-1}\|_{L_\infty(|x| \leqslant a_n(1+L\delta_n))}.
\end{aligned}$$

(b) For large $n \geqslant 1$ and $x \in I$, by (3.6)

$$\sum_{k \in [j(x)-2, j(x)+2] \cap [1,n]} l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) \lesssim 1.$$

Case 1: $0 < x < \frac{1}{2}$; By Lemma 3.3, (3.2),(3.3), and (3.5),

$$\begin{aligned}
&\sum_{k \notin [j(x)-2, j(x)+2] \cap [1,n]} l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) \\
&\lesssim |a_n^{1/2} p_n(x) w(x)|^2 \frac{a_n}{n} \sum \Psi_n(x_{kn}) (1 - |x_{kn}|/a_n + L\delta_n)^{1/2} \frac{\Delta x_{kn}}{|x - x_{kn}|^2} \\
&\lesssim \frac{a_n}{n} \sum \frac{\Delta x_{kn}}{|x - x_{kn}|^2} \lesssim \frac{1}{\Psi_n(x)} \lesssim 1.
\end{aligned}$$

Case 2: $\frac{1}{2} \leqslant x \leqslant a_{\frac{n}{2}}$; By (3.2), (3.3), and (3.5),

$$\begin{aligned}
&\sum_{k \notin [j(x)-2, j(x)+2] \cap [1,n]} l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) \\
&\quad \lesssim |a_n^{1/2} p_n(x) w(x)|^2 \frac{a_n}{n} \sum \Psi_n(x_{kn})(1 - |x_{kn}|/a_n + L\delta_n)^{1/2} \frac{\Delta x_{kn}}{|x - x_{kn}|^2} \\
&\quad \lesssim (1 - |x|/a_n)^{-1/2} \frac{a_n}{n} \sum \Psi_n(x_{kn})(1 - |x_{kn}|/a_n + L\delta_n)^{1/2} \frac{\Delta x_{kn}}{|x - x_{kn}|^2}.
\end{aligned}$$

Then

$$\sum \Psi_n(x_{kn})(1 - |x_{kn}|/a_n + L\delta_n)^{1/2} \frac{\Delta x_{kn}}{|x - x_{kn}|^2} := \sum_{|x_{kn}| < a_{\frac{2n}{3}}} + \sum_{|x_{kn}| \geqslant a_{\frac{2n}{3}}}$$

so we have by (2.4), (3.8), (3.10), and (3.11),

$$\sum_{|x_{kn}| \geqslant a_{\frac{2n}{3}}} \lesssim \frac{1}{T(a_n)} \frac{a_n(1 + L\delta_n) - a_{\frac{2n}{3}}}{(a_{\frac{2n}{3}} - a_{\frac{n}{2}})^2} \lesssim \frac{1}{a_n}$$

and by (2.4) and (3.9)

$$\sum_{|x_{kn}| < a_{\frac{2n}{3}}} \sim \sum (1 - |x_{kn}|/a_n + L\delta_n) \frac{\Delta x_{kn}}{|x - x_{kn}|^2} \lesssim \int_0^{x_*} + \int_{x^*}^{a_{\frac{2n}{3}}} \frac{1 - t/a_n}{(x - t)^2} dt,$$

where $x_* := x_{j(x)+1,n}$ and $x^* := x_{j(x)-1,n}$. Since by (3.9),

$$\int_{x^*}^{a_{\frac{2n}{3}}} \frac{1 - t/a_n}{(x - t)^2} dt \lesssim \frac{n(1 - x^*/a_n)}{a_n \Psi_n(x)} \sim \frac{n(1 - x/a_n)}{a_n(1 - x/a_n)^{1/2}} \lesssim \frac{n(1 - x/a_n)^{1/2}}{a_n}$$

and by integration by parts,

$$\begin{aligned}
\int_0^{x_*} \frac{1 - t/a_n}{(x - t)^2} dt &\lesssim \frac{1}{x} + \frac{n(1 - x_*/a_n)}{a_n \Psi_n(x)} + \left| \frac{1}{a_n} \int \frac{1}{x - t} dt \right| \\
&\lesssim 1 + \frac{n(1 - x/a_n)^{1/2}}{a_n} + \frac{\log n}{a_n},
\end{aligned}$$

we have

$$\sum (1 - |x_{kn}|/a_n + L\delta_n) \frac{\Delta x_{kn}}{|x - x_{kn}|^2} \lesssim 1 + \frac{n(1 - x/a_n)^{1/2}}{a_n} + \frac{\log n}{a_n}.$$

Therefore, for $\frac{1}{2} < x \leqslant a_{\frac{n}{2}}$ by (3.11) and(3.12),

$$\sum_{k \notin [j(x)-2, j(x)+2] \cap [1,n]} l_{kn}^2(x) w^2(x) w^{-2}(x_{kn})$$
$$\lesssim (1 - |x|/a_n)^{-1/2} \frac{a_n}{n} \left(1 + \frac{n(1 - x/a_n)^{1/2}}{a_n} + \frac{\log n}{a_n} \right)$$
$$\lesssim \frac{a_n (1 - |x|/a_n)^{-1/2}}{n} + 1 + \frac{\log n}{n} T^{1/2}(a_n)$$
$$\lesssim 1.$$

Case 3: $|x| \geqslant a_{\frac{n}{2}}$; By (3.2)–(3.5),

$$\sum_{k \notin [j(x)-2, j(x)+2] \cap [1,n]} l_{kn}^2(x) w^2(x) w^{-2}(x_{kn})$$
$$= (|1 - |x|/a_n| + L\delta_n)^{-1/2} \frac{a_n}{n} \sum \Psi_n(x_{kn})(1 - |x_{kn}|/a_n + L\delta_n)^{1/2} \frac{\Delta x_{kn}}{|x - x_{kn}|^2}$$
$$:= (|1 - |x|/a_n| + L\delta_n)^{-1/2} \left(\sum_{|x_{kn}| \leqslant a_{\frac{n}{3}}} + \sum_{|x_{kn}| \geqslant a_{\frac{n}{3}}} \right).$$

Then by (3.9)

$$\sum_{|x_{kn}| \leqslant a_{\frac{n}{3}}} \sim \frac{a_n}{n} \int_0^{a_{\frac{n}{3}}} \frac{1 - t/a_n}{(x-t)^2} dt \lesssim \frac{a_n}{n} \left(\frac{1 - t/a_n}{x - t} \bigg|_0^{a_{\frac{n}{3}}} + \frac{1}{a_n} \int_0^{a_{\frac{n}{3}}} \frac{dt}{x - t} \right)$$
$$\lesssim \frac{a_n}{n} \left(\frac{1 - a_{\frac{n}{3}}/a_n}{x - a_{\frac{n}{3}}} + \frac{1}{x} + \frac{\log a_n}{a_n} \right) \lesssim \frac{\log a_n}{n}$$

and by (2.4), (3.11), and Lemma 3.3,

$$\sum_{|x_{kn}| \geqslant a_{\frac{n}{3}}} \lesssim \frac{a_n}{nT(a_n)} \sum_{|x_{kn}| \geqslant a_{\frac{n}{3}}} \frac{\Delta x_{kn}}{|x - x_{kn}|^2}$$
$$\lesssim \frac{a_n}{nT(a_n)} \frac{n}{a_n \Psi_n(x)} \lesssim (|1 - |x|/a_n| + L\delta_n)^{1/2}.$$

Then, we have for $|x| \geqslant a_{\frac{n}{2}}$,

$$\frac{a_n}{n} \sum \Psi_n(x_{kn})(1 - |x_{kn}|/a_n + L\delta_n)^{1/2} \frac{\Delta x_{kn}}{|x - x_{kn}|^2}$$
$$\lesssim \frac{\log a_n}{n} + (|1 - |x|/a_n| + L\delta_n)^{1/2}.$$

Therefore, for $|x| \geqslant a_{\frac{n}{2}}$, by (3.11) and (3.12),

$$\sum_{k \notin [j(x)-2, j(x)+2] \cap [1,n]} l_{kn}^2(x) w^2(x) w^{-2}(x_{kn})$$

$$\lesssim |1 - |x|/a_n + L\delta_n|^{-1/2} \left(\frac{\log a_n}{n} + (|1 - |x|/a_n| + L\delta_n)^{1/2} \right)$$

$$\lesssim \frac{T^{\frac{1}{3}}(a_n) \log a_n}{n^{\frac{2}{3}}} + 1 \lesssim 1.$$

So we have, for all $x \in I$,

$$\sum_{k=1}^{n} l_{kn}^2(x) w^2(x) w^{-2}(x_{kn}) \lesssim 1.$$

(c) By (3.2)–(3.5), and Lemma 3.3,

$$\sum_{k \notin [j(x)-2, j(x)+2]} |x - x_{kn}| l_{kn}^2(x) w^{-2}(x_{kn}) w^2(x)$$

$$\lesssim |a_n^{1/2} p_n w(x)|^2 \frac{a_n}{n} \sum \frac{\Delta x_{kn}}{|x - x_{kn}|}$$

$$\lesssim |a_n^{1/2} p_n w(x)|^2 \frac{a_n \log n}{n}$$

$$\lesssim (|1 - |x|/a_n| + L\delta_n)^{-1/2} \frac{a_n \log n}{n}$$

and by (3.2) and (3.6)

$$\sum_{k \in [j(x)-2, j(x)+2]} |x - x_{kn}| l_{kn}^2(x) w^{-2}(x_{kn}) w^2(x)$$

$$\lesssim \sum_{k \in [j(x)-2, j(x)+2]} |x - x_{kn}| \lesssim \frac{a_n}{n} \Psi_n(x)$$

$$\lesssim \frac{a_n}{n} (|1 - |x|/a_n| + L\delta_n)^{-1/2}.$$

Therefore, we have for $x \in I$,

$$\sum |x - x_{kn}| l_{kn}^2(x) w^{-2}(x_{kn}) w^2(x)$$

$$\lesssim (|1 - |x|/a_n| + L\delta_n)^{-1/2} \frac{a_n \log n}{n}.$$

(d) From $\Phi_{\frac{a_n}{n}}(x) = |1-|x|/a_n|^{1/2} + T^{-1/2}(a_n)$, we have $\Phi^{-1}_{\frac{a_n}{n}}(x) \leqslant T^{1/2}(a_n)$ for $x \in I$. Then for $x \in I$ and by (c),

$$\begin{aligned}
&\sum_{k=1}^{n} \Phi^{-1}_{\frac{a_n}{n}}(x_{kn})|x-x_{kn}|l^2_{kn}(x)w^{-2}(x_{kn})w^2(x) \\
&\lesssim T^{1/2}(a_n) \sum_{k=1}^{n} |x-x_{kn}|l^2_{kn}(x)w^{-2}(x_{kn})w^2(x) \\
&\lesssim T^{1/2}(a_n)(|1-|x|/a_n| + L\delta_n)^{-1/2} \frac{a_n \log n}{n}.
\end{aligned}$$

On the other hand, we consider the case of $|x| \leqslant a_{\frac{n}{2}}$. Then (cf. (c))

$$\begin{aligned}
&\sum_{k \in [j(x)-2, j(x)+2]} \Phi^{-1}_{\frac{a_n}{n}}(x_{kn})|x-x_{kn}|l^2_{kn}(x)w^{-2}(x_{kn})w^2(x) \\
&\lesssim \Phi^{-1}_{\frac{a_n}{n}}(x) \sum_{k \in [j(x)-2, j(x)+2]} |x-x_{kn}|l^2_{kn}(x)w^{-2}(x_{kn})w^2(x) \\
&\lesssim \Phi^{-1}_{\frac{a_n}{n}}(x)(|1-|x|/a_n| + L\delta_n)^{-1/2} \frac{a_n \log n}{n}.
\end{aligned}$$

Since $\Phi_{\frac{a_n}{n}}(x) \sim (1-|x|/a_{2n})^{1/2}$ for $|x| \leqslant a_n(1+L\delta_n)$, we have (cf. (c))

$$\begin{aligned}
&\sum_{k \notin [j(x)-2, j(x)+2]} \Phi^{-1}_{\frac{a_n}{n}}(x_{kn})|x-x_{kn}|l^2_{kn}(x)w^{-2}(x_{kn})w^2(x) \\
&\lesssim |a_n^{1/2} p_n w(x)|^2 \frac{a_n}{n} \sum (1-|x_{kn}|/a_{2n})^{-1/2} \frac{\Delta x_{kn}}{|x-x_{kn}|} \\
&\lesssim (|1-|x|/a_n| + L\delta_n)^{-1/2} \frac{a_n}{n} \sum (1-|x_{kn}|/a_{2n})^{-1/2} \frac{\Delta x_{kn}}{|x-x_{kn}|}.
\end{aligned}$$

Here,

$$\begin{aligned}
&\sum_{k \notin [j(x)-2, j(x)+2]} (1-|x_{kn}|/a_{2n})^{-1/2} \frac{\Delta x_{kn}}{|x-x_{kn}|} \\
&\lesssim \int_0^{x_*} \frac{(1-t/a_{2n})^{-1/2}}{x-t}\,dt + \int_{x^*}^{\frac{x^*+a_n(1+L\delta_n)}{2}} \frac{(1-t/a_{2n})^{-1/2}}{t-x}\,dt \\
&\quad + \int_{\frac{x^*+a_n(1+L\delta_n)}{2}}^{a_n(1+L\delta_n)} \frac{(1-t/a_{2n})^{-1/2}}{t-x}\,dt
\end{aligned}$$

where $x_* \coloneqq x_{j(x)+1,n}$ and $x^* \coloneqq x_{j(x)-1,n}$. Then for $|x| \leqslant a_{\frac{n}{2}}$,

$$\int_0^{x_*} \frac{(1-t/a_{2n})^{-1/2}}{x-t}\,dt \lesssim (1-x_*/a_{2n})^{-1/2} \int_0^{x_*} \frac{1}{x-t}\,dt \lesssim (1-x/a_{2n})^{-1/2} \log n,$$

$$\begin{aligned}&\int_{x^*}^{\frac{x^*+a_n(1+L\delta_n)}{2}} \frac{(1-t/a_{2n})^{-1/2}}{t-x}\,dt \\ &\lesssim \left(1-\frac{x^*}{a_{2n}}+1-\frac{a_n(1+L\delta_n)}{a_{2n}}\right)^{-1/2} \int_{x^*}^{\frac{x^*+a_n(1+L\delta_n)}{2}} \frac{1}{t-x}\,dt \\ &\lesssim (1-x/a_{2n})^{-1/2} \log n\end{aligned}$$

and

$$\begin{aligned}&\int_{\frac{x^*+a_n(1+L\delta_n)}{2}}^{a_n(1+L\delta_n)} \frac{(1-t/a_{2n})^{-1/2}}{t-x}\,dt \\ &\lesssim \frac{1}{a_n(1+L\delta_n)-x} \int_{\frac{x^*+a_n(1+L\delta_n)}{2}}^{a_n(1+L\delta_n)} (1-t/a_{2n})^{-1/2}\,dt \\ &\lesssim \frac{-a_{2n}}{a_n(1+L\delta_n)-x} (1-t/a_{2n})^{1/2} \Big|_{\frac{x^*+a_n(1+L\delta_n)}{2}}^{a_n(1+L\delta_n)} \\ &\lesssim \frac{a_{2n}}{a_n(1+L\delta_n)-x}\left(1-\frac{x^*}{a_{2n}}+1-\frac{a_n(1+L\delta_n)}{a_{2n}}\right)^{1/2} \\ &\lesssim \frac{a_{2n}}{a_n}\left(1-\frac{x}{a_n}+L\delta_n\right)^{-1}\left(1-\frac{x^*}{a_{2n}}+1-\frac{a_n(1+L\delta_n)}{a_{2n}}\right)^{1/2} \\ &\sim (1-x/a_{2n})^{-1/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{k=1}^{n} \Phi_{\frac{a_n}{n}}^{-1}(x_{kn})|x-x_{kn}| l_{kn}^2(x) w^{-2}(x_{kn}) w^2(x) \\ &\lesssim (|1-|x|/a_n|+L\delta_n)^{-1/2} \frac{a_n \log n}{n} \begin{cases} (1-x/a_{2n})^{-1/2}, & |x| \leqslant a_{\frac{n}{2}} \\ T^{1/2}(a_n), & |x| \geqslant a_{\frac{n}{2}} \end{cases} \\ &\lesssim (|1-|x|/a_n|+L\delta_n)^{-1/2} \frac{a_n \log n}{n} \begin{cases} \Phi_{\frac{a_n}{n}}^{-1}(x), & |x| \leqslant a_{2n} \\ T^{1/2}(a_n), & |x| \geqslant a_{2n}. \end{cases}\end{aligned}$$

□

4. Proofs

Proof of Theorem 2.2. From (b) of Lemma 3.4,

$$\begin{aligned}||H_{n1}[f](x)w^2(x)||_{L_\infty(I)} &\lesssim ||fw^2||_{L_\infty(I)}\left\|\sum_{k=1}^{n} l_{kn}^2(x)w^{-2}(x_{kn})w^2(x)\right\|_{L_\infty(I)}\\ &\lesssim ||fw^2||_{L_\infty(I)}\end{aligned}$$

from Lemma 3.2 and (a) of Lemma 3.4,

$$\begin{aligned}&||H_{n2}[f](x)w^2v(x)||_{L_\infty(I)}\\ &\lesssim ||fw^2u||_{L_\infty(I)}\left\|\sum_{k=1}^{n} Q'(x_{kn})|x-x_{kn}|l_{kn}^2(x)w^{-2}(x_{kn})w^2(x)u^{-1}(x_{kn})v(x)\right\|_{L_\infty(I)}\\ &\lesssim ||fw^2u||_{L_\infty(I)}A_n\end{aligned}$$

and from (c) of Lemma 3.4,

$$\begin{aligned}&||H_{n3}[f'](x)w^2v(x)||_{L_\infty(I)}\\ &\lesssim ||f'w^2||_{L_\infty(I)}\left\|\sum_{k=1}^{n} |x-x_{kn}|l_{kn}^2(x)w^{-2}(x_{kn})w^2(x)v(x)\right\|_{L_\infty(I)}\\ &\lesssim ||f'w^2||_{L_\infty(I)}B_n.\end{aligned}$$

Therefore, we have (2.5), (2.6), and (2.8) and from (1.2) and (1.3), we have (2.7) and (2.9). □

Lemma 4.1. *Let $w\in\mathscr{A}$. For any polynomial $R\in\mathscr{P}_{2n-1}$ with n large enough, we have*

$$||(R(x)-H_n[R](x))w^2v(x)||_{L_\infty(I)}\lesssim ||R'w^2||_{L_\infty}B_n.$$

Proof. For any polynomial $R\in\mathscr{P}_{2n-1}$ and $x\in I$, from (c) of Lemma 3.4,

$$\begin{aligned}&|(R-H_n[R](x))w^2v(x)|\\ &\quad=\left|\sum_{k=1}^{n}(x-x_{kn})l_{kn}^2(x)R'(x_{kn})w^2(x)v(x)\right|\\ &\leqslant \sum |x-x_{kn}|l_{kn}^2(x)|R'(x_{kn})|w^2(x)v(x)\\ &\leqslant ||R'w^2||_{L_\infty}\sum |x-x_{kn}|l_{kn}^2(x)w^{-2}(x_{kn})w^2(x)v(x)\\ &\lesssim ||R'w^2||_{L_\infty}(|1-|x|/a_n|+L\delta_n)^{-1/2}v(x)\,\frac{a_n\log n}{n}.\end{aligned}$$

□

Proof of Theorem 2.3. For $x \in I$ and large enough n, by (2.3), (3.11), and (3.12),

$$
\begin{aligned}
&(|1-|x|/a_n| + L\delta_n)^{-1/2} v(x) \frac{a_n \log n}{n} \\
&\quad \lesssim \begin{cases} T(a_n)^{1/2} \frac{a_n \log n}{n}, & |x| \leqslant a_{n/2} \\ T^{1/3}(a_n)(\frac{a_n}{n})^{2/3} \log n, & |x| \geqslant a_{n/2} \end{cases} \\
&\quad \lesssim o(1)
\end{aligned}
$$

and

$$
\begin{aligned}
&(|1-|x|/a_n| + L\delta_n)^{-1/2} v(x) ||s_n(x)u^{-1}(x)||_{L_\infty(|x| \leqslant a_n(1+L\delta_n))} \frac{a_n \log n}{n} \\
&\quad \lesssim (|1-|x|/a_n| + L\delta_n)^{-1/2} v(x) \frac{a_n \log n}{n} = o(1).
\end{aligned}
$$

Therefore, for large enough $n \geqslant 1$, $A_n = o(1)$ and $B_n = o(1)$. For a given $\varepsilon > 0$, there exists a polynomial $R(x)$ such that (cf. [5, p. 180])

$$
|(f(x) - R(x)) w^2 u(x)| < \varepsilon.
$$

Then for large $n > 0$, we have from Lemma 4.1 and (2.7),

$$
\begin{aligned}
&||(f(x) - H_n[f](x)) w^2 v(x)||_{L_\infty(I)} \\
&\quad \leqslant ||(f(x) - R(x)) w^2 v(x)||_{L_\infty(I)} + ||(R - H_n[R](x)) w^2 v(x)||_{L_\infty(I)} \\
&\qquad + ||H_n[f - R](x) w^2 v(x)||_{L_\infty(I)} \\
&\quad \lesssim (A_n + 1) ||(f - R) w^2 u||_{L_\infty(I)} + ||R' w^2||_{L_\infty} B_n, \\
&\quad \lesssim \varepsilon + o(1), \qquad (4.1)
\end{aligned}
$$

from (1.3), (2.8), and (4.1),

$$
\begin{aligned}
&||(f(x) - \hat{H}_n[f](x)) w^2 v(x)||_{L_\infty(I)} \\
&\quad \leqslant ||(f(x) - H_n[f](x)) w^2 v(x)||_{L_\infty(I)} + ||\hat{H}_{n3}[f'](x) w^2 v(x)||_{L_\infty(I)} \\
&\quad \lesssim \varepsilon + o(1) + ||f' w^2||_{L_\infty} B_n \\
&\quad \lesssim \varepsilon + o(1),
\end{aligned}
$$

and from (2.5),

$$
\begin{aligned}
&||(f(x) - H_{n1}[f](x)) w^2(x)||_{L_\infty(I)} \\
&\quad \leqslant ||(f(x) - R(x)) w^2||_{L_\infty(I)} + ||H_{n1}[f - R](x) w^2(x)||_{L_\infty(I)} \\
&\quad \lesssim ||(f - R) w^2||_{L_\infty(I)} \\
&\quad \lesssim \varepsilon.
\end{aligned}
$$

Therefore, we have

$$\lim_{n\to\infty} ||(f(x)-H_n[f](x))w^2v(x)||_{L_\infty(I)} \lesssim \varepsilon,$$
$$\lim_{n\to\infty} ||(f(x)-\hat{H}_n[f](x))w^2v(x)||_{L_\infty(I)} \lesssim \varepsilon$$

and

$$\lim_{n\to\infty} ||(f(x)-H_{n1}[f](x))w^2v(x)||_{L_\infty(I)} \lesssim \varepsilon.$$

Since ε is arbitrary, we have the results. □

Proof of Theorem 2.4. Let R be any polynomial of degree $\leqslant 2n-1$. From (d) in Lemma 3.4, for $x \in I$

$$\begin{aligned}
&|(R(x)-H_n[R](x))w^2(x)| \\
&\quad = \left|\sum_{k=1}^{n}(x-x_{kn})l_{kn}^2(x)R'(x_{kn})w^2(x)\right| \\
&\quad \leqslant \left\| R'(x)\Phi_{\frac{a_n}{n}}(x)w^2(x)\right\|_{L_\infty} \sum |x-x_{kn}|\Phi_{\frac{a_n}{n}}^{-1}(x_{kn})l_{kn}^2(x)w^{-2}(x_{kn})w^2(x) \\
&\quad \lesssim \frac{a_n}{n}||R'\Phi_{\frac{a_n}{n}}w^2||_{L_\infty}(|1-|x|/a_n|+L\delta_n)^{-1/2}\begin{cases}\Phi_{\frac{a_n}{n}}^{-1}(x), & |x|\leqslant a_{2n},\\ T^{1/2}(a_n), & |x|\geqslant a_{2n}.\end{cases}
\end{aligned}$$

Since for $|x| \leqslant a_n(1+L\delta_n)$, by (2.4),(3.7), and (3.14),

$$\begin{aligned}
|s_n(x)| &= |Q'(x)\Psi_n(x)|1-|x|/a_n+L\delta_n|^{1/2}| \\
&\lesssim \begin{cases} Q'(x)(1-|x|/a_n), & |x|\leqslant a_{\frac{n}{2}}\\ Q'(x)\frac{1}{T(a_n)}, & a_{\frac{n}{2}}\leqslant |x|\leqslant a_n(1+L\delta_n)\end{cases} \\
&\lesssim \frac{n}{a_n}\Phi_{\frac{a_n}{n}}(x) \lesssim \frac{n}{a_n},
\end{aligned}$$

we have for $x \in I$,

$$\begin{aligned}
&|H_n[f-R](x)w^2(x)| \lesssim ||(f-R)w^2||_{L_\infty(I)} \\
&\quad + ||(f-R)w^2||_{L_\infty(I)}(|1-|x|/a_n|+L\delta_n)^{-1/2}||s_n(x)||_{L_\infty(|x|\leqslant a_n(1+L\delta_n))}\frac{a_n\log n}{n} \\
&\quad \lesssim ||(f-R)w^2||_{L_\infty(I)}(1+(|1-|x|/a_n|+L\delta_n)^{-1/2}\log n) \\
&\quad \lesssim ||(f-R)w^2||_{L_\infty(I)}\left(1+(|1-|x|/a_n|+L\delta_n)^{-1/2}\log n\begin{cases}\Phi_{\frac{a_n}{n}}^{-1}(x), & |x|\leqslant a_{2n}\\ T^{1/2}(a_n), & |x|\geqslant a_{2n}\end{cases}\right).
\end{aligned}$$

Therefore, we have for $x\in I$

$$
\begin{aligned}
&|(f(x)-H_n[f](x))w^2(x)|\\
&\quad\leqslant|(f(x)-R(x))w^2(x)|+|(R-H_n[R](x))w^2(x)|+|H_n[f-R](x)w^2(x)|\\
&\quad\lesssim\left(||(f-R)w^2||_{L_\infty(I)}+\frac{a_n}{n}||R'\Phi_{\frac{a_n}{n}}w^2||_{L_\infty}\right)\\
&\qquad\times\left(1+(|1-|x|/a_n|+L\delta_n)^{-1/2}\log n\begin{cases}\Phi_{\frac{a_n}{n}}^{-1}(x), & |x|\leqslant a_{2n}\\ T^{1/2}(a_n), & |x|\geqslant a_{2n}\end{cases}\right)\\
&\quad\lesssim K_{1,2n-1,\infty}\left(f,w^2,\frac{a_n}{n}\right)\\
&\qquad\times\left(1+(|1-|x|/a_n|+L\delta_n)^{-1/2}\log n\begin{cases}\Phi_{\frac{a_n}{n}}^{-1}(x), & |x|\leqslant a_{2n}\\ T^{1/2}(a_n), & |x|\geqslant a_{2n}\end{cases}\right). \qquad\square
\end{aligned}
$$

Proof of Corollary 2.5.

$$
\begin{aligned}
|I[k;f]-I_n[k,f]|&\leqslant\int_I|k(x)(f-H_n[f])|\,dx\\
&\leqslant||(f-H_n[f])w^2v(x)||_{L_\infty(I)}\int_I|k(x)|w^{-2}(x)v^{-1}(x)\,dx\\
&\lesssim||(f-H_n[f])w^2v(x)||_{L_\infty(I)}.
\end{aligned}
$$

By Theorem 2.3, we have the result for $I_n[k;f]$ and by the same process, we have the results for $\hat{I}_n[k;f]$ and $J_n[k;f]$. $\square$

5. Applications: $\mathscr{F}$, $\mathscr{E}_1$, and $\mathscr{EXP}$ cases

5.1. Freud weight case

We consider Freud weights $w:=\exp(-Q)$ where $Q:\mathbb{R}\to\mathbb{R}$ is of polynomial growth at infinity.

Definition 5.1. Freud class $\mathscr{F}$: Let $w\in\mathscr{A}$ on $I=\mathbb{R}$ and suppose that there exist $A>1$ and $B>1$ such that

$$A\leqslant T(x)\leqslant B, x\in I+. \tag{5.1}$$

Then we write $w\in\mathscr{F}$.

A typical example of the Freud weights is

$$w_\alpha(x) := \exp(-|x|^\alpha), \quad \alpha > 1, \ x \in \mathbb{R}.$$

Specially, the case of $\alpha = 2$ is the Hermite weight.

Proposition 5.2. *Let $w \in \mathscr{F}$. If A and B are the same as in* (5.1), *then*

$$u^{1/B} \leqslant a_u/a_1 \leqslant u^{1/A}, \quad u \in [1, \infty). \tag{5.2}$$

Proof. This is Lemma 5.2(b) [8, p. 478].

Corollary 5.3. *Let $w \in \mathscr{F}$. Let $u(x) := (1 + |Q'(x)|)$ and $v(x) = (1 + |x|)^{-(1-\frac{2\alpha}{3})}$ with $\alpha < \min\{A, 3/2\}$, where A is the same as in* (5.1). *Then for a continuous function f on $\mathbb{R}$ with* (2.11) *we have*

$$\lim_{n\to\infty} ||(f(x) - H_n[w^2, f](x))w^2 v(x)|| = 0$$

and for a continuous function f on $\mathbb{R}$ satisfying (2.11) *and $||f' w^2|| < \infty$,*

$$\lim_{n\to\infty} ||(f(x) - \hat{H}_n[w^2, f](x))w^2 v(x)||_{L_\infty(I)} = 0.$$

Moreover, (2.12) *and* (2.13) *hold under these assumptions.*

Proof. Since for $|x| \leqslant a_n(1 + L\delta_n)$, by (2.4),

$$|s_n(x)u^{-1}(x)| \sim \Psi_n(x)|1 - |x|/a_n + L\delta_n|^{1/2} \lesssim 1$$

and for $x \in \mathbb{R}$ and $0 < \beta < 1$, by (3.7), (5.1), and (5.2),

$$\begin{aligned}
&(|1 - |x|/a_n| + L\delta_n)^{-1/2} v(x) \frac{a_n \log n}{n} \\
&\quad = \frac{(|1 - |x|/a_n| + L\delta_n)^{-1/2}}{(1 + |x|)^{1-\frac{2\alpha}{3}}} \frac{a_n \log n}{n} \\
&\quad \lesssim \begin{cases} \frac{a_n \log n}{n} & |x| \leqslant a_{\beta n}, \\ (\frac{a_n^\alpha}{n})^{\frac{2}{3}} \log n & |x| \geqslant a_{\beta n}, \end{cases} \\
&\quad \lesssim \begin{cases} n^{\frac{1}{A}-1} \log n & |x| \leqslant a_{\beta n}, \\ n^{(\frac{\alpha}{A}-1)\frac{2}{3}} \log n & |x| \geqslant a_{\beta n} \end{cases} = o(1),
\end{aligned}$$

we have $A_n = o(1)$ and $B_n = o(1)$ for large enough n. Then we have the results by the same process as in the proofs of Theorem 2.3 and Corollary 2.5. □

5.2. Erdös weight case

We consider Erdös weights $w := \exp(-Q)$ where $Q : \mathbb{R} \to \mathbb{R}$ is even and is of faster than polynomial growth at infinity.

Definition 5.4. Erdös class $\mathscr{E}_1$: Let $w \in \mathscr{A}$ on $I = \mathbb{R}$ and suppose that $T(x)$ is increasing in $I+ = (0, \infty)$ with

$$\lim_{x \to \infty} T(x) = \infty, \quad T(0+) := \lim_{x \to 0+} T(x) > 1.$$

Moreover, if we have

$$T(x) \leqslant C(Q(x))^{\varepsilon}, \quad x \to \infty \tag{5.3}$$

for some positive constant C independent of x, then we write $w \in \mathscr{E}_1$.

The archetypal examples of $w \in \mathscr{E}_1$ are

(1) $$w_{k,\alpha}(x) := \exp(-\exp_k(|x|^{\alpha})), \quad x \in \mathbb{R},$$

where $\alpha > 0$, k is a positive integer, and $\exp_k(\,) = \exp(\exp(\exp(\cdots)))$ denotes the kth iterated exponential.

(2) $$w_{A,B}(x) := \exp(-\exp(\log(A + x^2)^B), \quad x \in \mathbb{R},$$

where A is a fixed but large enough positive number and $B > 1$.

For example for $w_{k,\alpha}$,

$$T(x) = T_{k,\alpha}(x) = \alpha\left[1 + x^{\alpha} \sum_{l=1}^{k} \prod_{j=1}^{l-1} \exp_j(x^{\alpha})\right], \quad x \geqslant 0$$

and so (2.1) and (5.3) hold in the stronger form,

$$\lim_{|x| \to \infty \text{ or } 1} T(x) \Big/ \left(\frac{xQ'(x)}{Q(x)}\right) = 1; \quad \lim_{x \to \infty} \frac{T(x)}{[\prod_{j=1}^{k} \log_j Q(x)]} = \alpha.$$

Proposition 5.5. *Let $w \in \mathscr{E}_1$. Let $\varepsilon > 0$. Then*

$$a_n \leqslant Cn^{\varepsilon} \quad \text{and} \quad T(a_n) \leqslant Cn^{\varepsilon}, \ n \geqslant 1. \tag{5.4}$$

Proof. This is Lemma 2.4 in [4].

Corollary 5.6. *Let $w \in \mathscr{E}_1$. Let $u(x) := Q'(x)^{\Delta}$ with $\Delta > \frac{1}{3}$ and $v(x) = 1$. Then for a continuous function f on $\mathbb{R}$ with* (2.11) *we have*

$$\lim_{n \to \infty} ||(f(x) - H_n[w^2, f](x))w^2 v(x)|| = 0$$

and for a continuous function f *on* $\mathbb{R}$ *satisfying* (2.11) *and* $\|f'w^2\| < \infty$,

$$\lim_{n\to\infty} \|(f(x) - \hat{H}_n[w^2, f](x))w^2 v(x)\|_{L_\infty(I)} = 0.$$

Moreover, (2.12) *and* (2.13) *hold under these assumptions.*

Proof. Since for $|x| \leqslant a_n(1 + L\delta_n)$, by (2.4), (3.7), and (5.4),

$$|s_n(x)u^{-1}(x)| \sim Q'(x)^{1-\Delta}\Psi_n(x)|1 - |x|/a_n + L\delta_n|^{1/2} \lesssim \left(\frac{n}{a_n}\right)^{1-\Delta} T^{\frac{1}{3}}(a_n) \lesssim n^{\frac{1-\Delta}{2}}$$

and for $x \in \mathbb{R}$ and $0 < \beta < 1$, by (3.10) and(5.4),

$$(|1 - |x|/a_n| + L\delta_n)^{-1/2} \frac{a_n \log n}{n} \lesssim \begin{cases} \frac{a_n T^{1/2}(a_n) \log n}{n}, & |x| \leqslant a_{\beta n} \\ \frac{a_n T^{\frac{1}{3}}(a_n) \log n}{n^{\frac{2}{3}}}, & |x| \geqslant a_{\beta n} \end{cases} \lesssim \frac{a_n T^{\frac{1}{3}}(a_n) \log n}{n^{\frac{2}{3}}} \lesssim n^{-1/3},$$

we have $A_n \lesssim n^{\frac{1}{2}(\frac{1}{3} - \Delta)} = o(1)$ and $B_n \lesssim n^{-\frac{1}{3}} = o(1)$ for large $n \geqslant 1$. Then we have the results by the same process as in the proofs of Theorem 2.3 and Corollary 2.5. $\square$

5.3. Exponential weight on $(-1, 1)$ *case*

Definition 5.7. Class of exponential Weights on $(-1, 1)$ $\mathscr{EXP}$: Let $w \in \mathscr{A}$ on $I = (-1, 1)$ and suppose that $T(x)$ is increasing in $(0, 1)$ with

$$\lim_{x\to 0+} T(x) > 1$$

and that for some $A > 2$ and x close enough to 1,

$$T(x) \geqslant \frac{A}{1 - x^2}.$$

Then we write $w \in \mathscr{EXP}$.

The archetypal examples of $w \in \mathscr{EXP}$ are

$$w_{0,\alpha}(x) := \exp(-(1 - x^2)^{-\alpha})), \ \alpha > 0, \ x \in (-1, 1)$$

or

$$w_{k,\alpha}(x) := \exp(-\exp_k(1 - x^2)^{-\alpha})), \ k \geqslant 1, \ \alpha > 0, x \in (-1, 1).$$

Corollary 5.8. *Let* $w \in \mathscr{EXP}$. *Let* $u(x) := (1 + |Q'(x)|)^{\Delta} T^{1/3}(x)$ *with* $\Delta > \frac{1}{3}$ *and* $v(x) = T^{-1/3}(x)$. *Then for a continuous function* f *on* $(-1, 1)$ *with* (2.11) *we have*

$$\lim_{n \to \infty} ||(f(x) - H_n[w^2, f](x))w^2 v(x)|| = 0$$

and for a continuous function f *on* $(-1, 1)$ *satisfying* (2.11) *and* $||f' w^2|| < \infty$,

$$\lim_{n \to \infty} ||(f(x) - \hat{H}_n[w^2, f](x))w^2 v(x)||_{L_\infty(I)} = 0.$$

Moreover, (2.12) *and* (2.13) *hold under these assumptions.*

Proof. Since for $|x| \leqslant a_n(1 + L\delta_n)$, by (2.4), (3.7), and $a_n \sim 1$,

$$|s_n(x)u^{-1}(x)| \sim Q'(x)^{1-\Delta} T^{-\frac{1}{3}}(x) \Psi_n(x) |1 - |x|/a_n + L\delta_n|^{1/2} \lesssim n^{1-\Delta}$$

and for $x \in (-1, 1)$ and $0 < \beta < 1$, by (3.10)–(3.12),

$$(|1 - |x|/a_n + L\delta_n)^{-1/2} v(x) \frac{a_n \log n}{n} = (|1 - |x|/a_n + L\delta_n)^{-1/2} T^{-\frac{1}{3}}(x) \frac{a_n \log n}{n} \lesssim \begin{cases} \frac{T^{\frac{1}{6}}(a_n) \log n}{n} & |x| \leqslant a_{\beta n}, \\ \frac{\log n}{n^{\frac{2}{3}}} & |x| \geqslant a_{\beta n}, \end{cases} \lesssim \frac{\log n}{n^{\frac{2}{3}}},$$

we have $A_n \lesssim n^{(\frac{1}{3} - \Delta)} \log n = o(1)$ and $B_n \lesssim n^{-\frac{2}{3}} \log n = o(1)$ for large $n \geqslant 1$. Then we have the results by the same process as in the proofs of Theorem 2.3 and Corollary 2.5. □

Acknowledgments

The author thanks the referee for many valuable comments and corrections. This work was supported by Korea Research Foundation Grant (KRF-2002-050-C00003).

References

[1] S.B. Damelin, The weighted Lebesgue constant of Lagrange interpolation for exponential weights on $[-1, 1]$, Acta Math. (Hungarica) 81 (3) (1998) 211–228.
[2] S.B. Damelin, The Lebesgue function and Lebesgue constant of Lagrange interpolation for Erdös weights, J. Approx. Theory 94 (2) (1998) 235–262.

[3] S.B. Damelin, H.S. Jung, K.H. Kwon, On mean convergence of Hermite–Fejer and Hermite interpolation for Erdös weights, J. Comput. Appl. Math. 137 (2) (2001) 71–76.
[4] S.B. Damelin, D.S. Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdös weights, Canad. J. Math. 48 (4) (1996) 710–736.
[5] Z. Ditzian, V. Totik, Moduli of Smoothness, in: Springer Series in Computational Mathematics, Vol. 9, Springer, Berlin, 1987.
[6] G. Freud, Orthogonal Polynomials, Pergamon Press, Oxford, 1971.
[7] H.S. Jung, K.H. Kwon, Mean convergence of Hermite–Fejer and Hermite interpolation for Freud weights, J. Comput. Appl. Math. 99 (1998) 219–238.
[8] A.L. Levin, D.S. Lubinsky, Christoffel Functions, Orthogonal Polynomials and Nevai's conjecture for Freud weights, Constr. Approx. 8 (1992) 461–533.
[9] A.L. Levin, D.S. Lubinsky, Christoffel functions and orthogonal polynomials for Exponential weights on $[-1,1]$, Mem. Amer. Math. Soc. 535 (1994).
[10] A.L. Levin, D.S. Lubinsky, T.Z. Mthembu, Christoffel Functions and Orthogonal Polynomials for Erdös weights on $(-\infty,\infty)$, Rend. Mat. Appl. 14 (7) (1994) 199–289.
[11] D.S. Lubinsky, Hermite and Hermite–Fejér interpolation and associated product integration rules on the real line: L_∞ theory, J. Approx. Theory 70 (1992) 284–334.
[12] D.S. Lubinsky, The weighted L_p-norms of orthogonal polynomials for Erdös weights, Comput. Math. Appl. 33 (1997) 151–163.
[13] D.S. Lubinsky, F. Moricz, The weighted L_p-norms of orthogonal polynomials for Freud weights, J. Approx. Theory 77 (1994) 42–50.
[14] D.S. Lubinsky, P. Rabinowitz, Hermite and Hermite–Fejér interpolation and associated product integration rules on the real line: L_1 theory, Canad. J. Math. 44 (1992) 561–590.
[15] J. Szabados, Weighted Lagrange interpolation and Hermite–Fejer interpolation on the real line, J. Ineq. Appl. 1 (1997) 99–123.
[16] G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. 23 (1975).